Math 261B Thurs. Oct. 22

$$V^\lambda = H^0(G/B, \mathcal{L}_{w_0(\lambda)})$$

GL_n $\quad G/B = \{ 0 \subset F_1 \subset F_2 \cdots \subset F_{n-1} \subset K^n \mid \dim F_d = d \}$

$E_\bullet \quad E_d = \langle e_1, \dots, e_d \rangle$ has $\operatorname{stab}_{GL_n} E_\bullet = B$

$\mathcal{L} = \Lambda^d K^n / F_{n-d}$ has fiber $\Lambda^d K^n / E_{n-d}$

basis vector $e_{n-d+1} \wedge \cdots \wedge e_n$

weight $(0 \cdots 0\, 1^d)$

$\mathcal{L} = \mathcal{L}_{(0 \cdots 0\, 1^d)} = \mathcal{L}_{w_0(\lambda)}$ $\qquad W = S_n$

$\lambda = (1^d\, 0 \cdots 0)$ $\qquad w_0 = \begin{matrix} 1 & \cdots & n \\ & \downarrow & \\ n & \cdots & 1 \end{matrix}$

$V = \Lambda^d K^n$ has basis of weight vectors

$e_{i_1} \wedge \cdots \wedge e_{i_d}$ $\qquad (0\, 1 \cdots 0\, 1)$ d 1's

Only dominant weight is $\lambda = (1^d\, 0 \cdots 0)$

$\Rightarrow V$ is V^λ

$H^0(G/B, \Lambda^d(K^n/F_{n-d})) \cong \Lambda^d K^n = V$

map $V \to \mathfrak{g}$ is $\Lambda^d K^n \longrightarrow \Lambda^d K^n / F_{n-d}$

$Gr^n_{n-d} = \{ F \subset K^n \mid \dim F = n-d \}$ depends only on F_{n-d}

G/B $\quad F_\bullet \mapsto F_{n-d}$ $\quad E_\bullet \mapsto E_{n-d}$ $\rightsquigarrow$ Stabilizer

$$\text{Parabolic subgroup } P = \begin{array}{c|c} & \begin{array}{cc} & \end{array} \\ \begin{array}{c} n-d \\ d \end{array} & \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \\ & \begin{array}{cc} n-d & d \end{array} \end{array}$$

$P = B W_J B \qquad W_J \subset W = S_n \qquad W_J = S_{n-d} \times S_d \subset S_n$

$J = \{1, \dots, n-1\} \setminus \{n-d\}$

$G = \coprod_{w \in W} BwB$ (LPU factorization for $G = GL_n$)

(Bruhat decomposition)

Subgroup gen. by S_i $\quad i \neq n-d$

$= \text{Stabilizer}_W(w_0(\lambda))$ $\quad \langle \alpha_i^\vee, w_0(\lambda) \rangle = 0$

General set-up:

$V^\lambda \supset (V^\lambda)_\lambda$ 1-dim'l subspace, fixed by B,

$p = (V^\lambda)_\lambda \in \mathbb{P}(V^\lambda)$

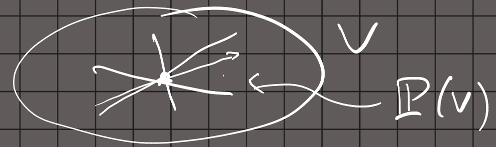
λ regular: $\mathrm{Stab}_W(\lambda) = 1$

$J = \emptyset$, $W_J = 1$, $P = B$

full stabilizer is some $P \supset B$: $P = B W_J B$ $\quad W_J = \mathrm{Stab}_W(\lambda)$

$G/B \twoheadrightarrow G/P \hookrightarrow \mathbb{P}(V^\lambda)$

$gP \mapsto gp$



Tautological line bundle $\mathcal{L}$ on $\mathbb{P}(V^\lambda)$ is $\mathcal{L}_\lambda$. Its dual

$\mathcal{L}_{-\lambda} \cong \mathcal{L}_\lambda^*$ has $H^0(\mathbb{P}(V^\lambda), \mathcal{O}(1)) = (V^\lambda)^*$

$\mathcal{L}_\lambda^* = \mathcal{O}(1)$

$(V^\lambda)^* \xrightarrow{\sim} H^0(G/P, \mathcal{L}_{-\lambda})$ $\qquad P = B W_J B$

$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad W_J = \mathrm{Stab}(\lambda).$

$V^\mu \cong H^0(G/P, \mathcal{L}_{w_0(\mu)})$

$\mu = w_0(-\lambda)$

$= -w_0(\lambda)$

$\mathrm{Stab}_W(\lambda) = \mathrm{Stab}_W(w_0(\mu))$

$\mathbb{P}((V^\mu)^*) =$

$\mathbb{P}((K^n)^*) = \mathrm{Gr}^n_{n-1}(K^n) \ni F \qquad K^n/F$

Another GL_n example $\qquad \lambda = (d, 0, \ldots, 0)$

$\mathcal{L} = \mathcal{L}_{w_0(\lambda)} = \mathcal{L}_{(0,\ldots,0,d)} = \left(K^n/F_{n-1}\right)^{\otimes d}$

comes from $G/P = Gr^n_{n-1} = \mathbb{P}^{n-1} \quad \left(= \mathbb{P}((K^n)^*) \right)$

$\mathcal{L}$ is $\mathcal{O}(d)$

$H^0(G/P, \mathcal{L}_{w_0(\lambda)}) = H^0(\mathbb{P}^{n-1}, \mathcal{O}(d)) = S^d K^n$

$= K[x_1, \ldots, x_n]_d$

coordinates on $(K^n)^*$

$g \cdot (x_1, \ldots, x_n)$

$= (x_1, \ldots, x_n) \begin{pmatrix} \text{matrix} \\ g \end{pmatrix}$

$\underline{x}^{\underline{\nu}} = x_1^{\nu_1} \cdots x_n^{\nu_n}$ has weight $\underline{\nu}$, giving $(|\underline{\nu}| = d)$ a basis of weight vectors in $S^d K^n$

Highest weight is $\nu = (d, 0, \ldots, 0)$ $\qquad x_1^d$

There are plenty of dominant ν in here

Root subgroup U_α for $\alpha = e_i - e_j$

$$c \mapsto \begin{array}{c|c} & \\ i & \begin{pmatrix} 1 & & c \\ & \ddots & \\ & & 1 \end{pmatrix} \\ & \quad\;\; j \end{array} \qquad x_j \mapsto x_j + c x_i$$

U invariant $f(x)$ fixed by for $i<j$.

x_1^d is the only one in degree d.

$$\Delta z = z \otimes z$$
$$\Delta a_{ij} = \sum_k a_{ik} \otimes a_{kj}$$

$\mathbb{Z}$ forms.

$$\mathcal{O}_K(GL_n) = K[a_{11}, \dots, a_{nn}, z] / (\det(A)z - 1)$$
$$\mathcal{O}_{\mathbb{Z}}(GL_n) = \mathbb{Z}[a_{11}, \dots, a_{nn}, z] / (\det(A)z - 1)$$

still a Hopf algebra $\quad \Delta : \mathcal{O}_{\mathbb{Z}}(G) \to \mathcal{O}_{\mathbb{Z}}(G) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}(G)$

$$K \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{Z}}(GL_n) = \mathcal{O}_K(GL_n)$$

$$GL_n(K) = (K\text{-alg homs} : \mathcal{O}_K(GL_n) \xrightarrow[e_g]{} K)$$

$$\begin{tikzcd}
\mathcal{O}_k(G) \arrow[r, "\Delta"] \arrow[ddr, bend right, "e_{gh}"'] & \mathcal{O}_k(G) \otimes_k \mathcal{O}_k(G) \arrow[d, "e_g \otimes e_h"] \\
& K \otimes_k K \arrow[d, equal] \\
& k
\end{tikzcd}$$

$$(\text{Ring homs} : \mathcal{O}_{\mathbb{Z}}(GL_n) \to R) = (\operatorname{Spec}(R) \to \operatorname{Spec} \mathcal{O}_{\mathbb{Z}}(GL_n))$$
$$\| $$
$$GL_n(R)$$
$$\|$$
invertible $n \times n$ matrices over R.

$$PGL_n(K) = GL_n(K) / k \cdot 1$$

$Sp_{2n}(K)$: matrices preserving $x^T J_- y$ $\quad \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$

$$A^T J_- A = J_-$$

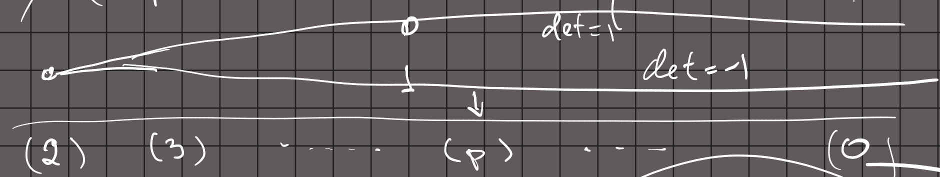
or $\quad -J_- A^T J_- \cdot A = I$ ⟵ polynomial eq'ns in the matrix entries ⟵ over $\mathbb{Z}$

$$\rightsquigarrow \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \mapsto \left(\begin{array}{c|c} -d^R & b^R \\ \hline c^R & -a^R \end{array}\right)$$

$$\mathcal{O}_{\mathbb{Z}}(Sp_{2n}) = \mathbb{Z}[a_{11}, \dots, a_{2n,2n}] / (\text{these eq'ns}).$$

$\mathcal{O}_{\mathbb{Z}}(O_N) = \mathbb{Z}[\cdots] / (\text{preserve standard quadratic form})$

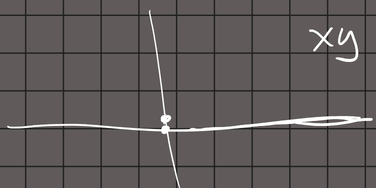
SO_{2n} is trickier



Spec $\mathcal{O}_{\mathbb{Z}}(O_{2n})$

Spec$(\mathbb{Z})$

$$\frac{1-\det}{2} = 0$$



$xy = 0$

$y^2 = 0$